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FUNDAMENTALS OF THE AUTOMATIC CONTROL THEORY

CALCULATION WORK

Recommended by the Methodical Council of Igor Sikorsky KPI
as a textbook for students
who study in the specialty 171 "Electronics"
educational scientific program "Electronic components and systems"

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«FUNDAMENTALS OF THE AUTOMATIC CONTROL THEORY»

CALCULATION WORK

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The discipline "Fundamentals of the automatic control theory" is an integral part of the disciplines that are included in the list of mandatory disciplines of engineering training in electronic systems. The aim of the discipline is to study the basic principles of calculation of closed automatic control systems. In the process of studying the course, students acquire knowledge of the basics of the description of closed continuous, discrete and nonlinear systems and methods of their calculation. The manual contains guidelines for performing calculation and graphic work on "Analysis of the properties and operation of automatic control systems." The manual is intended for students majoring in 171 Electronics, educational scientific program Electronic Components and Systems.

Content

Introduction

The aim of these guidelines is to assist students in self-study of the basics of the theory of automatic control and the implementation of computational and graphic work on the course.

As a result of studying the theoretical provisions of the discipline and self-control tasks execution in the main sections, students must learn:

- how to compose equations that describe the processes in electronic devices;
- ho to find transfer functions;
- stability, how to calculate transients and steady-state processes in continuous, pulse, linear and nonlinear automatic control systems.

Solving tasks contributes to the practical consolidation of the material of the main sections of the discipline.

Calculation and graphic work consists of 7 separate tasks. All tasks are provided with guidelines, theoretical information, explanations, examples of solutions, drawings and graphs that facilitate the implementation of tasks and learning material.

When studying the materials of the discipline, special attention should be paid to the method of compiling differential equations and their solution, based on the use of Laplace transform, as similar approaches with minor changes are used to describe and study electromagnetic processes in power devices and electronic systems.

Theoretical provisions of automatic control for systems with delay, pulse systems are widely used in the analysis of transients and steady-state processes in open and closed electronics systems. It is necessary to study the methods of studying the stability of linear (continuous and pulsed) and nonlinear electronic devices, as these questions often arise when calculating closed electronic systems.

Before performing computational and graphic work, it is desirable for students to repeat the properties of continuous and discrete Laplace transform, Fourier transform, theoretical foundations of electrical engineering and methods of electronic circuits calculation.

The numbers of variants of tasks of calculation and graphic work correspond to the last two digits of the number of the record book – Nb. If it is more than 12, then the option number is calculated by the formula

$$Nv=(Nb-12)*2+3,$$

where Nb – last two digits of the number of the record book.

All tasks use the coefficient K, which is determined by the first a_1, the second a_2 digits and the record book number:

$$K= a_2*Nb+10a_1$$

For example, for record book number 9116:

- a_1=9;
- a_2=1;
- Nb=16;
- variant number Nv=(Nb-12)*2+3=(16-12)*2+3=7;
- K= a_2*Nb+10a_1= 1*16+10*9=106.

<u>Task 1.</u> Find the operator transfer function and time characteristics for a passive quadrupole. Determine the reaction of the `circuit to the setting influence g(t). Numbers of variants with the corresponding numbers of figures (fig. 1-12) and parameters of elements of schemes are given in tab.1.

Guidelines

While preparing the task 1 you need to get familiar with:

- the method of compiling differential and operator equations for linear electric circuits, as well as with the matrix form of their notation [1, 5];
- with the relation of differential equations, transfer functions and time characteristics [1, 5];
- with the Laplace transform and its properties, formulas (theorems) of decomposition [1].

<u>Instruction!</u> Transfer function W(p) you first need to bring in general:

$$W(p) = \frac{B(p)}{A(p)} = \frac{b_m p^m + b_{m-1} p^{m-1} + ... + b_1 p + b_0}{a_m p^m + a_{m-1} p^{m-1} + ... + a_1 p + a_0},$$

and then substitute the numerical values of the parameters and lead to a normalized form:

$$W(p) = K_0 \frac{B_0(p)}{A_0(p)},$$

where $K_0 = b_0 / a_0$; $\mathrm{B}_0(p) = \frac{b_m p^m}{b_0} + \frac{b_{m-1} p^{m-1}}{b_0} + ... + \frac{b_1 p}{b_0} + 1$;

$$A_0 = \frac{a_m p^m}{a_0} + \frac{a_{m-1} p^{m-1}}{a_0} + ... + \frac{a_1 p}{a_0} + 1.$$

Time characteristics of the circuit – weight (pulse) ω(t) and transient h(t) functions are found by the transfer function W (p):

$$\omega(t) = L^{-1}\{W(p)\}$$

$$h(t) = L^{-1}\{\frac{W(p)}{p}\},$$

where L^{-1} is the symbol of the inverse Laplace transform.

Decomposition formulas are used to find the originals ω(t) and h(t).

Table 1

Variant number	**Figure number**	**R_1, Ohm**	**R_2, Ohm**	**L_1, mH**	**L_2, mH**	**C_1, uF**	**C_2, uF**	**g(t)**
1	1	2K	100	15K	-	10K	-	δ (t-5)
2	2	50	-	5K	-	1000	-	sin10t
3	3	100	-	10K	-	10	5K	5t
4	4	20K	150	5K	-	10K	-	$10e^{-2t}$
5	5	5K	120	-	-	K	10K	δ (t)
6	6	0,1K	200	-	-	5K	15K	cos50t
7	7	100	5K	K	-	20K	-	10
8	8	K	2K	10K	-	1000	-	sin5t
9	9	50	100	5K	20K	-	-	δ (t-2)
10	10	100	15K	10	-	50K	-	$5+10e^{-10t}$
11	11	500	10	15K	-	50	-	10cost
12	12	0,5K	30	10	-	15K	-	5(t-2)

Transfer functions are, as a rule, fractional-rational functions of the complex variable p, i.e. they can be represented as

$$W(p) = \frac{B(p)}{A(p)}.$$

If the fraction $\frac{B(p)}{A(p)}$ is not reduced, i.e. the degree of the polynomial of the denominator A(p) is greater than the degree of the polynomial of the numerator B(p), the original of the function W(p) can be found by the roots of the characteristic polynomial A(p), using the ***residue theorem***. The zeros of the denominator (roots of the characteristic polynomial) are called the ***poles*** of the function W(p). Consider two cases:

1) all poles are simple
2) all or some poles are multiples.

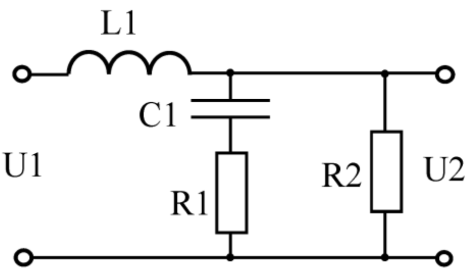

Fig. 1

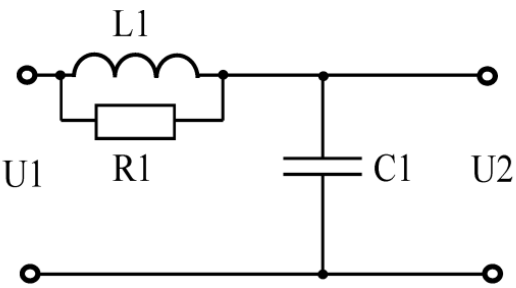

Fig. 2

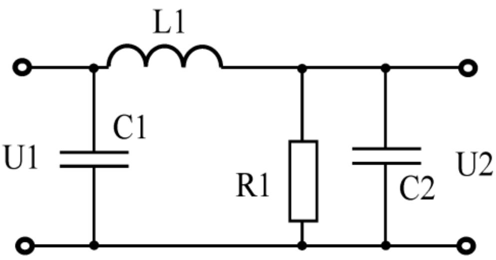

Fig. 3

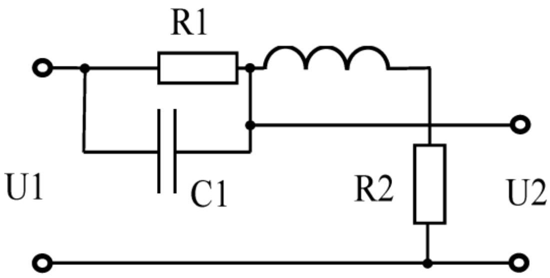

Fig. 4

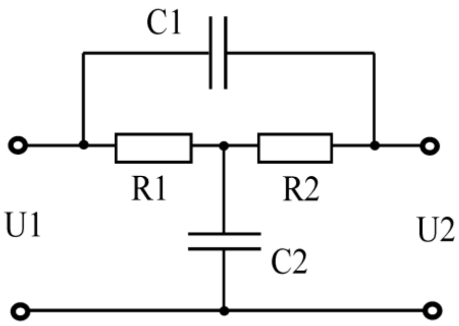

Fig. 5

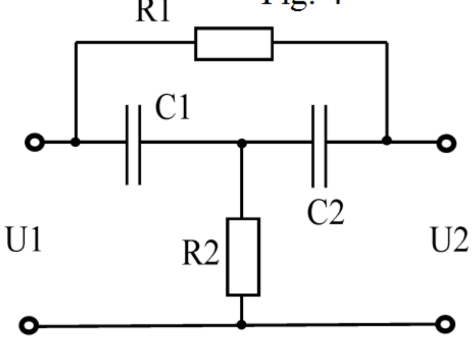

Fig. 6

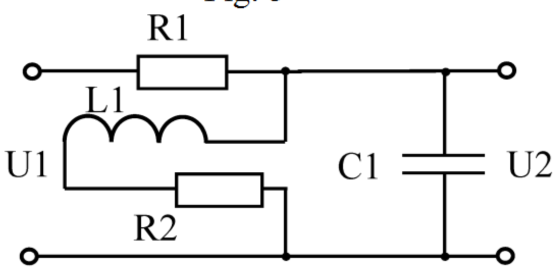

Fig. 7

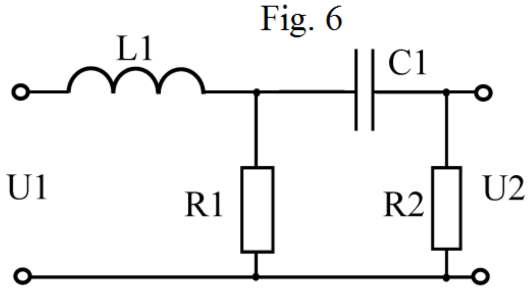

Fig. 8

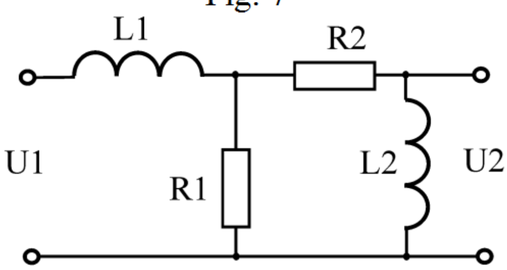

Fig. 9

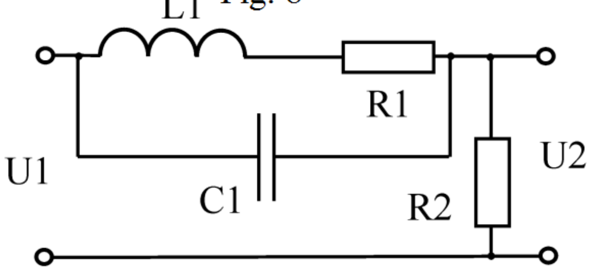

Fig. 10

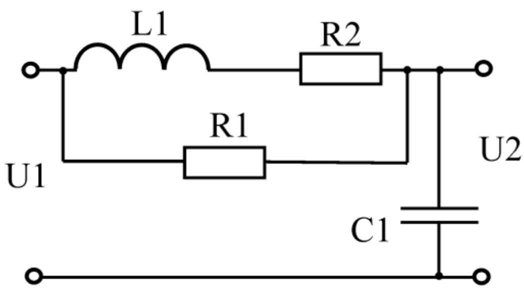

Fig. 11

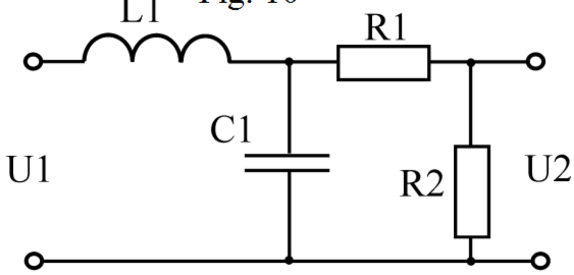

Fig. 12

In the first case, the original function is determined by the expression:

$$\omega(t)=\sum_{k=1}^{n}\operatorname{Re}s\left[W(p)e^{pt},p_k\right]=\sum_{k=1}^{n}\frac{B(p_k)}{A'(p_k)}e^{p_k t},$$

where n – is the degree of polynomial A(p); p_k – are the roots of the characteristic equation A(p)=0;

$$A'(p_k)=\left.\frac{dA(p)}{dp}\right|_{p=p_k}.$$

If the function W(p) has a zero pole, that is $W(p)=\frac{B(p)}{pC(p)}$, then

$$\omega(t)=\frac{B(0)}{C(0)}+\sum_{k=2}^{n}\frac{B(p_k)}{p_kC'(p_k)}e^{p_k t}$$

If the function W(p) has complex-conjugate poles $p=\alpha\pm j\beta$, then the residue functions at these points will be complex-conjugate, so their sum is equal to twice the real part:

$$\begin{aligned}&\operatorname{Re}s[W(p),\alpha+j\beta]+\operatorname{Re}s[W(p),\alpha-j\beta]=2\operatorname{Re}\{\operatorname{Re}s[W(p),\alpha+j\beta]\}=\\&=2\operatorname{Re}\{\operatorname{Re}s[W(p),\alpha-j\beta]\}\end{aligned}.$$

In the second case (multiple roots), the original function is determined by the equation:

$$\omega(t)=\sum_{k=1}^{n}\frac{1}{(m_k-1)!}\lim_{p\to p_k}\left\{\frac{d^{m_k-1}}{dp^{m_k-1}}\left[\left[p-p_k\right]^{m_k}\frac{B(p)}{A(p)}e^{pt}\right]\right\},$$

де m_1 – is the multiplicity of the k-th pole.

Remark. In cases when the order of the numerator of the image W(p) is equal to or higher than that of the denominator, when the inverse transformation of the original function ω(t) appears δ(t)-function and its derivative, for example:

$$W(p)=\frac{p+\alpha}{p+\beta}=1+\frac{\alpha-\beta}{p+\beta}.$$

Applying the inverse Laplace transform, we obtain:

$$\omega(t)=L^{-1}\{W(p)\}\ =\delta(\mathrm{t})+(\alpha-\beta)\mathrm{e}^{-\beta \mathrm{t}}.$$

Example. Derive the operator transfer function of the passive quadrupole, the scheme of which is shown in Fig. 13. Find its time characteristics at the following values of parameters: R1 = 50 Ohm, R2 = 150 Ohm, R3 = 300 Ohm, L = 0,1 H, C=10^{-3} F.

In the study of complex electronic systems, the equations are mainly written in matrix form, and one of the simplest and most convenient for the analysis of electronic circuits is a system of nodal equations (the method of nodal voltages). The transfer function of the system in this case can be calculated directly from the conduction matrix of the circuit

$$W(p) = \frac{\Delta(\alpha+\gamma)(\beta+\delta)}{\Delta(\alpha+\gamma)(\alpha+\gamma) + G_l \Delta(\alpha+\gamma)(\alpha+\gamma),(\beta+\delta)(\beta+\delta)},$$

where $\Delta(\alpha+\gamma)(\beta+\delta)$ – is the total algebraic complement obtained from the determinant Δ of the scheme matrix after deleting the row α and adding it to the row γ, deleting the column β and adding it to the column δ, as well as multiplying by $(-1)^{\alpha+\beta}$; γ, α and β, δ – are the input and output nodes of the quadrupole respectively, G_l – is the load conductivity.

If γ or δ coincides with the base node, i.e. γ = 0 or δ = 0, then the addition with index 0 is reduced to a single index, which means the exclusion of the corresponding row or column.

In practice, there is often a case where the input and output of the quadrupole have a common node, which is chosen as the base. With this

$$W(p) = \frac{\Delta\alpha\beta}{\Delta\alpha\alpha + G_l + \Delta\alpha\alpha\beta\beta}.$$

To find the transfer function of the quadrupole (see Fig. 13) it is necessary to pre-form a conduction matrix of the circuit. For this example, it looks like:

$$Y = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & G_1 & -G_1 & 0 & 0 \\ 2 & -G_1 & G_1 + G_2 + \frac{1}{pL} & -G_2 & 0 \\ 3 & 0 & -G_2 & G_2 + G_3 + pC_1 & -G_3 \\ 4 & 0 & 0 & -G_3 & G_3 \end{array}$$

Find the transfer function using the ratio

$$W(p) = \frac{\Delta_{14}}{\Delta_{11}},$$

	1	2	3
2	$-G_1$	$G_1 + G_2 + \frac{1}{pL}$	$-G_2$
3	0	$-G_2$	$G_2 + G_3 + pC_1$
4	0	0	$-G_3$

$$\Delta_{14} = \; = G_1 G_2 G_3$$

	2	3	4	
2	$G_1 + G_2 + \frac{1}{pL}$	$-G_2$	0	
3	$-G_2$	$G_2 + G_3 + pC_1$	$-G_3$	=
4	0	$-G_3$	G_3	

$$\Delta_{11} = \begin{aligned} &= (-1)^{1+1}\left[\left(G_1 + G_2 + \frac{1}{pL}\right)(G_2 + G_3 + pC_1)G_3 - G_3^2\left(G_1 + G_2 + \frac{1}{pL}\right) - G_2^2 G_3\right] = \\ &= \frac{p^2 L_1 C_1 (G_1 G_3 + G_2 G_3) + p(C_1 G_3 + L_1 G_1 G_2 G_3) + G_2 G_3}{pL_1}, \end{aligned}$$

then

$$W(p) = \frac{pL_1 G_1 G_2}{p^2 L_1 C_1 (G_1 + G_2) + p(C_1 + L_1 G_1 G_2) + G_2}.$$

Substituting the numerical values of the parameters

$$W(p) = \frac{p \cdot 0.1 \cdot 2 \cdot 10^{-2} \cdot 6.667 \cdot 10^{-3}}{p^2 \cdot 0.1 \cdot 10^{-2} \cdot 2.667 \cdot 10^{-3} + 1.01333 \cdot 10^{-3} p + 6.667 \cdot 10^{-3}},$$

and leading to a normalized form, we eventually get

$$W(p) = 2 \cdot 10^{-3} \frac{p}{4 \cdot 10^{-4} p^2 + 0.152 p + 1}.$$

Let`s find the time characteristics of the schemes. Let`s find the roots of the characteristic polynomial (W(p) poles)

$$4\cdot10^{-4}p^2+0.152p+1=0.$$

We will receive: $p_1=-6.696975;\ p_2=-373.3030$.

To find the weight function we use the decomposition formula (the case of simple poles):

$$\omega(t)=\sum_{k=1}^{2}\frac{B(p_k)}{A`(p_k)}e^{p_k t}=2\cdot10^{-3}\left[\frac{-6.696975e^{-6.696975t}}{8\cdot10^{-4}(-6.696975)+0.152}+\frac{-373.303e^{-373.303t}}{8\cdot10^{-4}(-373.303)+0.152}\right]=$$
$$=5.091337e^{-6.696975t}-0.091337e^{-373.303t}.$$

Let`s find the transient characteristic

$$h(t)=L^{-1}\{W(p)/p\}=L^{-1}\{H(p)\};$$

$$H(p)=\frac{2\cdot10^{-3}}{4\cdot10^{-4}p^2+0.125p+1}.$$

Then

$$h(t)=2\cdot10^{-3}\left[\frac{e^{-6.696975t}}{8\cdot10^{-4}(-6.696975)+0.152}+\frac{e^{-373.303t}}{8\cdot10^{-4}(-373.303)+0.152}\right]$$
$$=0.01364e^{-6.696975t}-0.01364e^{-373.303t}.$$

$$h(t)=2\cdot10^{-3}\left[\frac{e^{-6.696975t}}{8\cdot10^{-4}(-6.696975)+0.152}+\frac{e^{-373.303t}}{8\cdot10^{-4}(-373.303)+0.152}\right]$$
$$=0.01364e^{-6.696975t}-0.01364e^{-373.303t}.$$

Let`s find the reaction of the circuit to the input influence $\mathrm{g(t)=5+\sin 2t}$.

Image of the input signal is

$$\mathrm{G(p)=L\{g(t)\}=\frac{5}{p}+\frac{2}{p^2+4}}.$$

Image of the output signal is

$$X(p)=W(p)G(p)=2\cdot10^{-3}\frac{p}{4\cdot10^{-4}p^2+0.152p+1}\left[\frac{5}{p}+\frac{2}{p^2+4}\right].$$

The output signal as a function of time is found by the decomposition equations. Let`s find the poles of the function X(p):

$$p(p^2+4)(4\cdot10^{-4}p^2+0{,}152p+1)=0$$
$$p_1=0;\quad \mathrm{p}_{2,3}=\pm j2;\quad \mathrm{p}_4=-6{,}696975;\quad \mathrm{p}_5=-373{,}303.$$

Then

$$x(t) = 2 \cdot 10^{-3} \frac{-6,696975(5(44,85+4)-13,39)}{4,02-182,63+134,55-8,14+4} e^{-6,696975} +$$

$$+2 \cdot 10^{-3} \frac{-373,303(5(13,9 \cdot 10^4 + 4) - 746,606)}{38,84 \cdot 10^9 - 31,63 \cdot 10^6 + 13,94 \cdot 10^4 - 453,94 + 4} e^{-373,303} =$$

$$= 6,41 \cdot 10^{-2} e^{-6,696975} - 1,33 \cdot 10^{-5} e^{-373,303}.$$

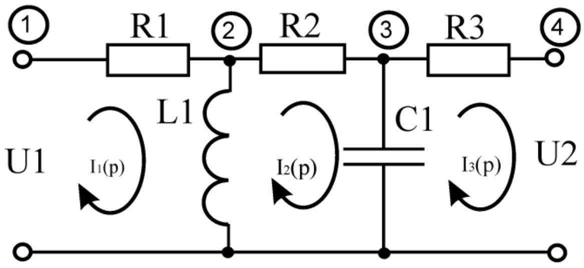

Fig. 13

<u>Task 2.</u> Build the Nyquist plot, frequency response and phase response for the system with transfer function W(p). The initial data are given in table 2.

Guidelines

While preparing the task 2 you need to:

1. Learn the different types of frequency characteristics, methods of their building [6].
2. Be familiar with the frequency characteristics of typical links[2, 6].

While building the phase response it is necessary to consider that

$$\phi(\omega) = \begin{cases} arctg \dfrac{V(\omega)}{U(\omega)}, & U(\omega) \geq 0; \\ \pi + \mathrm{arctg} \dfrac{V(\omega)}{U(\omega)}, & U(\omega) < 0, V(\omega) \geq 0; \\ -\pi + \mathrm{arctg} \dfrac{V(\omega)}{U(\omega)}, & U(\omega) < 0, V(\omega) < 0, \end{cases}$$

where $U(\omega)$, $V(\omega)$ – real and imaginary parts of Nyquist plot, respectively.

Transfer functions of aperiodic link of the second order and the oscillating link in appearance coincide:

$$W(p)=\frac{K}{T_2^2p^2+T_1p+1}.$$

However, for the aperiodic link of the second order $T_1 \geq T_2$ (the roots of the characteristic equation are real). The transfer function of the aperiodic link of the second order can be represented as:

$$W(p)=\frac{K}{(T_3p+1)(T_4p+1)},$$

where

$$T_{3,4}=\frac{T_1}{2}\pm\sqrt{\left(\frac{T_1}{2}\right)^2-T_2^2}.$$

Table 2

Variant number	W(p)	К$_1$	T$_1$, sec	T$_2$, sec	T$_3$, sec	a	b
1	K К$_1$p/[(T $_1$p+1)(T $_2$p-1)]	10	0.1	0.01	-	-	-
2	K К$_1$p/[(T $_1$p+1)(T $_2$p-1)]	0.1	1	100	-	-	-
3	K К$_1$(T $_1$p+1)/[p(T $_2$p-1)]	10	0.1	10	-	-	-
4	K К$_1$p^2/[(a p^2+ в p+1)]	100	-	-	-	1	4
5	K К$_1$(T $_1$p-1)/[p^2(T $_2$p+1)]	50	100	0.01	-	-	-
6	K К$_1$/[p(a p^2+ в p+1)]	0.01	-	-	-	0.1	0.2
7	K К$_1$p/(T $_1$p+1)^2	10	100	-	-	-	-
8	K К$_1$(T $_1$p-1)/[p^2 (T $_2$p+1)]	20	0.1	10	-	-	-
9	К$_1$(T $_1$p-1)/(T $_2$p+1)	10	0.01	10K	-	-	-
10	К$_1$/[(T $_1$p-1) (a p^2+ в p+1)]	100	0.1	-	-	K	5
11	K К$_1$/[(T $_1$p-1)(T $_2$p-1)]	5	0.1K	100	-	-	-
12	K К$_1$p/[(T $_1$p-1)(T $_2$p-1)]	0.5	0.01K	10	-	-	-

<u>Example.</u> Build the Nyquist plot, frequency response and phase response for the system with transfer function $W(p)=\frac{Kp}{Tp+1}$, where K=100; T=0,1.

We distinguish the real and imaginary parts of the complex transfer function:

$$W(j\omega) = W(p)\big|_{p=j\omega} = \frac{Kj\omega}{Tj\omega + 1} = \frac{Kj\omega(1 - Tj\omega)}{(1 + Tj\omega)(1 - Tj\omega)} = \frac{\omega^2 KT + j\omega K}{1 + \omega^2 T^2}.$$

Thus, $U(\omega) = \dfrac{\omega^2 KT}{1 + \omega^2 T^2}$, $V(\omega) = \dfrac{\omega K}{1 + \omega^2 T^2}$. We define the module and the argument of the function $W(j\omega)$:

$$W(\omega) = \sqrt{U(\omega)^2 + V(\omega)^2} = \frac{\omega K}{1 + \omega^2 T^2},$$

$$\varphi(\omega) = arctg\,\frac{1}{\omega T}.$$

Given the value of the frequency ω from zero to infinity, we determine the numerical values of the functions $W(\omega)$, $U(\omega)$, $V(\omega)$, $\varphi(\omega)$ and summarize them in table. 3. Frequency characteristics points are determined by boundary transitions:

$$U(0) = \lim_{\omega \to 0} \frac{\omega^2 KT}{1 + \omega^2 T^2} = 0,\ V(0) = \lim_{\omega \to 0} \frac{\omega K}{1 + \omega^2 T^2} = 0,\ W(0) = \lim_{\omega \to 0} \frac{\omega K}{1 + \omega^2 T^2} = 0,$$

$$\varphi(0) = \lim_{\omega \to 0} arctg\,\frac{1}{\omega T} = \frac{\pi}{2};$$

$$U(\infty) = \lim_{\omega \to \infty} \frac{\omega^2 KT}{1 + \omega^2 T^2} = \frac{K}{T},\ V(\infty) = \lim_{\omega \to \infty} \frac{\omega K}{1 + \omega^2 T^2} = 0,\ W(\infty) = \lim_{\omega \to \infty} \frac{\omega K}{1 + \omega^2 T^2} = \frac{K}{T},$$

$$\varphi(\infty) = \lim_{\omega \to \infty} arctg\,\frac{1}{\omega T} = 0.$$

Graphs of functions are shown on Fig.14-16.

While building the phase response of complex links and systems, their transfer functions are also appropriate to represent as the product of simpler links. With this, the phase response of the system is defined as the sum of the phase responses of individual links. For given example: $\varphi_1(\omega) = 0$, $\varphi_2(\omega) = \dfrac{\pi}{2}$, $\varphi_3(\omega) = -arctgT$. By adding these characteristics, we also obtain the phase response of the system (Fig. 15).

Таблиця 3

ω	U(ω)	V(ω)	W(ω)	φ(ω)
0	0	0	0	
1	99	99	99,5	1,47
3	82,57	275,23	287,35	1,28
5	200	400	447,21	1,11
7	328,86	469,8	573,46	0,96
10	500	500	707,11	0,785
15	692	461	832,05	0,588
20	800	400	894,43	0,464
100	990,1	99,01	995,04	0,0997
1000	999,9	9,999	999,95	0,0099
100000	1000	0,1	1000	0,001

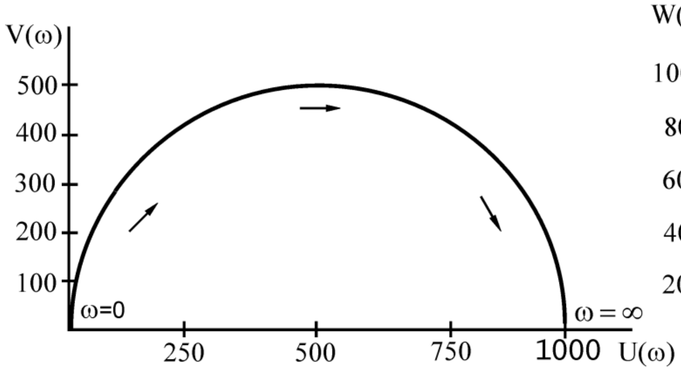

Fig. 14

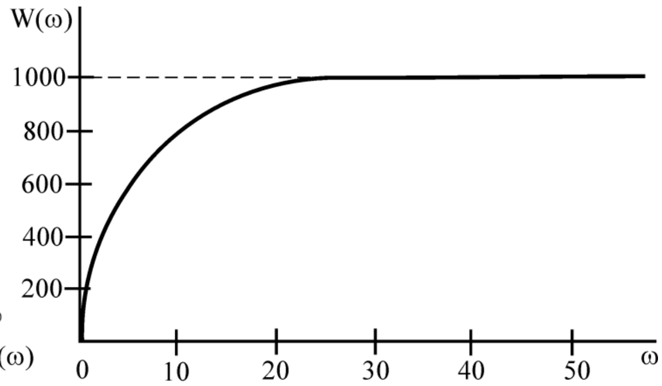

Fig. 16

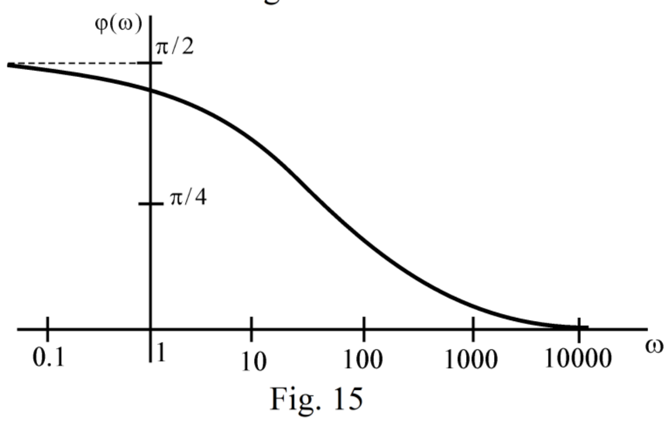

Fig. 15

Task 3. Explore the stability of the system covered by the feedback loop by the following methods:

1) Routh criterion (Fig. 17), W(p) corresponds to transfer function, defined in Task 1, $W_0(p) = \frac{K}{p^2 + 0.1Kp + 2K}$;

2) Hurwitz criterion (same conditions);

3) Mikhailov criterion (same conditions);

4) Nyquist criterion (see Fig. 17) W(p) corresponds to transfer function, defined in Task 2, $W_0(p)=1$.

Guidelines

While preparing the task 3 you need to:

1. Be familiar with the concept of "stability" for linear continuous automatic control systems, a necessary and sufficient condition of stability [2, 4].

2. To study the algebraic and frequency criteria of stability of linear automatic control systems, as well as to analyze solutions to problems on this topic [2, 3].

When studying the stability using the criteria of Routh, Hurwitz, Mikhailov, it is necessary to pre-determine the transfer function of a closed system

$$W_{closed}(p) = \frac{W(p)}{1 \pm W_0(p)W(p)} = \frac{b_m p^m + ... + b_1 p + b_0}{a_n p^n + ... a_1 p + a_0} = \frac{B(p)}{A(p)},$$

where sign "+" is for systems with negative feedback, "-" is for systems with positive feedback.

Algebraic stability criteria (Routh, Hurwitz) allows to determine the stability of the system by the coefficients of the characteristic polynomial

$$A(p) = a_n p^n + a_{n-1} p^{n-1} + ... a_1 p + a_0 .$$

Routh stability criterion. In order for the automatic control system to be stable, it is necessary and sufficient that the coefficients of the first column of the Rouse table (table 4) have the same sign, i.e. when $a_n > 0$ were positive.

Table 4

Coefficient r_i	Row i	Column K		
		1	2	3
-	1	$a_n = c_{11}$	$a_{n-2} = c_{21}$	$a_{n-4} = c_{31}$
-	2	$a_{n-1} = c_{12}$	$a_{n-3} = c_{22}$	$a_{n-5} = c_{32}$
$r_3 = \frac{c_{11}}{c_{12}}$	3	$c_{13} = c_{21} - r_3 c_{22}$	$c_{23} = c_{31} - r_3 c_{32}$	$c_{33} = c_{41} - r_3 c_{42}$
$r_4 = \frac{c_{12}}{c_{13}}$	4	$c_{14} = c_{22} - r_4 c_{23}$	$c_{24} = c_{32} - r_4 c_{33}$	$c_{34} = c_{42} - r_4 c_{43}$
$r_5 = \frac{c_{13}}{c_{14}}$	5	$c_{15} = c_{23} - r_5 c_{24}$	$c_{25} = c_{33} - r_5 c_{34}$	$c_{35} = c_{43} - r_5 c_{44}$
……….	……..	……………………	……………………..	………………
$r_i = \frac{c_{1,i-2}}{c_{1,i-1}}$		$c_{1,i} = c_{2,i-2} - r_i c_{2,i-1}$	$c_{2,i} = c_{3,i-2} - r_i c_{3,i-1}$	$c_{3,i} = c_{4,i-2} - r_i c_{4,i-1}$
……….	……..	……………………	……………………..	………………

Thus, the first two rows of the Routh table are filled with the coefficients of the characteristic equation. Any of the other coefficients in this table are determined in accordance with the ratio

$$c_{K,i} = c_{K+1,i-2} - r_i c_{K+1,i-1};$$
$$r_i = \frac{c_{1,i-2}}{c_{1,i-1}},$$

where K , i – is the number of a column and row respectively.

The number of rows in the Routh table is equal to the power of the characteristic equation plus one (n +1).

If not all the coefficients of the first column are positive, then the system is unstable, and the number of roots of the characteristic polynomial lying in the right half-plane (instability index) is equal to the number of sign changes in the 1st column.

Example. Investigate the stability of the system (see Fig. 17) using the Routh criterion:

$$W(p) = \frac{10}{(p+1)(10p+1)}; \; W_0(p) = \frac{5}{p^2 + 10p + 2}.$$

Let’s define the transfer function of a closed system

$$W_{closed} = \frac{\frac{10}{(p+1)(10p+1)}}{1+\frac{10}{(p+1)(10p+1)}\frac{5}{p^2+10p+2}} = \frac{10\left(p^2+10p+2\right)}{10p^4+111p^3+131p^2+32p+52}.$$

Characteristic polynomial of the system $A(p) = 10p^4 + 111p^3 + 131p^2 + 32p + 52.$

Let's make the Routh table (table 5):

Table 5

Coefficient r_i	Row i	Column K 1	Column K 2	Column K 3
-	1	$a_4 = 10 = c_{11}$	$a_2 = 131 = c_{21}$	$a_0 = 52 = c_{31}$
-	2	$a_3 = 111 = c_{12}$	$a_1 = 32 = c_{22}$	0
$r_3 = \frac{a_4}{a_3} = 0.09$	3	$c_{13} = c_{21} - r_3 c_{22} = = 131 - 0.9 \cdot 32 = 128.12$	$c_{23} = c_{31} - r_3 c_{32} = 52 - - 0.09 \cdot 0 = 52$	0
$r_4 = \frac{a_3}{c_{13}} = = \frac{111}{128.12} = 0.866$	4	$c_{14} = c_{22} - r_4 c_{23} = = 32 - 0.866 \cdot 52 = -13.05$	0	0
$r_5 = \frac{c_{13}}{c_{14}} = = \frac{128.12}{-13.05} = -9.818$	5	$c_{15} = c_{23} - r_5 c_{24} = = 52 - (-9.818) \cdot 0 = 52$	0	0

There are two changes in the sign of the coefficients of the first column, so the system is unstable, and the characteristic equation has two right roots.

Hurwitz stability criterion. For the stability of a linear system it is necessary and sufficient that when $a_n > 0$ n main determinants of the following matrix of the characteristic equation of this system were positive

$$\Delta n = \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \ldots\ldots & 0 \\ a_n & a_{n-2} & a_{n-4} & a_{n-6} & \ldots\ldots. & 0 \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \ldots\ldots. & 0 \\ 0 & a_n & a_{n-2} & a_{n-4} & \ldots\ldots. & 0 \\ \ldots\ldots.. & \ldots\ldots... & \ldots\ldots.. & \ldots\ldots... & \ldots\ldots. & \ldots\ldots \\ 0 & 0 & 0 & \ldots\ldots.. & a_1 & 0 \\ 0 & 0 & 0 & \ldots\ldots.. & a_2 & a_0 \end{vmatrix}$$

i.e. $a_n > 0$; $\Delta_1 = a_{n-1} > 0$; $\Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} > 0 \ldots \Delta n > 0$.

Example. Investigate the stability of the system using the Hurwitz criterion (initial data are the same as in the previous example).

Characteristic polynomial of the system

$$A(p) = 10p^4 + 111p^3 + 131p^2 + 32p + 52 .$$

Let’s define the Hurwitz’s determinant

$$\Delta_4 = \begin{vmatrix} 111 & 32 & 0 & 0 \\ 10 & 132 & 52 & 0 \\ 0 & 111 & 32 & 0 \\ 0 & 10 & 132 & 52 \end{vmatrix} .$$

In accordance with the Hurwitz stability criterion we obtain

$$a_4 = 10 > 0; \quad \Delta_1 = a_3 = 111 > 0; \quad \Delta_2 = \begin{vmatrix} 111 & 32 \\ 10 & 132 \end{vmatrix} = 14332 > 0;$$

$$\Delta_3 = \begin{vmatrix} 111 & 32 & 0 \\ 10 & 132 & 52 \\ 0 & 111 & 32 \end{vmatrix} = 111 \cdot 132 \cdot 32 - 111^2 \cdot 52 - 32^2 \cdot 10 = -182068 < 0.$$

Thus, this system is <u>unstable</u>.

Mikhailov stability criterion. This is one of the frequency stability criteria, which allows us to judge the stability of the system on the basis of some curve, Mikhailov curve (hodograph). To build it, it is necessary to substitute $j\omega$ in the characteristic polynomial instead of p, select the real and imaginary parts of $A(j\omega)=P(j\omega)+Q(j\omega)$.

By changing the frequency from 0 to ∞, a Mikhailov curve is built in the plane of parameters $P(j\omega)$ and $Q(j\omega)$.

In order for a linear system to be stable, it is necessary and sufficient that the vector of the Mikhailov curve when changing the frequency from zero to infinity returned, nowhere to zero, around the origin counterclockwise at an angle $n\frac{\pi}{2}$, where n is the order of the characteristic polynomial, i.e. the Mikhailov hodograph when the frequency changes from zero to infinity, starting at ω = 0 on a significant positive semiaxis, it bypassed only against the time arrow consecutively n quadrants of the coordinate plane (all the time surrounding the origin). On Fig. 18 typical Mikhailov curves for stable systems starting from the first (n = 1) and ending with the fifth (n = 5) order are shown.

Example. Investigate the stability of the system using the Mikhailov criterion (initial data are the same as in the previous example).

Characteristic polynomial

$$A(p) = 10p^4 + 111p^3 + 131p^2 + 32p + 52.$$

Instead of p we substitute jω and define real and imaginary part

$$A(j\omega) = 10\omega^4 - j111\omega^3 - 132\omega^2 - j32\omega + 52;$$
$$P(\omega) = 10\omega^4 - 132\omega^2 + 52 \;;\quad Q(\omega) = -111\omega^3 - 32\omega.$$

By changing the frequency from zero to infinity, we determine (Table 6) and build a Mikhailov hodograph (Fig. 19). Since the Mikhailov hodograph does not pass sequentially n = 4 quadrants counterclockwise, surrounding the origin, the system is unstable.

Table 6

ω	0	0.2	0.5	0.55	0.65	0.7	1	3	5	100
P(ω)	52	46.74	19.63	12.98	-1.96	-10.28	-70	-326	3002	$9.9*10^8$
Q(ω)	0	5.51	2.13	-0.87	-9.68	-15.68	-79	-2901	-13715	$-1.1*10^8$

Nyquist stability criterion. This criterion for the type of Nyquist plot of an open system allows us to make conclusion on the stability of a closed system.

If the open circuit of the system is stable or neutral, then for the stability of a closed system it is necessary and sufficient that the Nyquist plot of the open circuit does not surround the point (-1, j0) (Fig. 20).

If the system is unstable in the open state, then for the stability of a closed system it is necessary and sufficient that the Nyquist plot of the open circle surrounds the point (-1, j0) counterclockwise at an angle π, where L is the number of roots of the characteristic polynomial of the open circuit, that have positive significant parts (instability index). For example, if the instability index L = 1, then for the stability of a closed system the Nyquist plot of an open circuit should look approximately as shown on Fig. 21.

With a complex form of the characteristic $W_{open}(j\omega)$ there may be difficulties in determining the number of its revolutions around the point (-1, j0). In this case, it is advisable to use the "transition rule" to assess stability. The transition of the characteristic through the weight axis to the left of the point (-1, j0), i.e. through the segment $[-\infty,-1]$, with increasing frequency is called positive if it is carried out from top to bottom, and negative – if it is from bottom to top. If the characteristic $W_{open}(j\omega)$ begins or ends on the segment $[-\infty,-1]$, it is considered that it makes a half-transition (Fig. 22). Then the Nyquist criterion can be formulated as follows: in order for a closed system to be stable, it is necessary and sufficient that the difference between the number of positive and negative transitions of the Nyquist plot of the open system $W_{open}(j\omega)$ through the segment of the weight axis $[-\infty,-1]$ when changing the frequency from zero to infinity is equal to L/2, where L is the number of right roots of the characteristic equation of an open system, the number of positive transitions to the left of the point (-1, j0) should be equal to the number of negative transitions.

If the transfer function of the open circuit of the system has astatism of the ν–th order (has ν zero poles)

$$W_{open}(p) = \frac{b_m p^m + ... + b_1 p + b_0}{p^{\nu}(a_n p^n + ... + a_1 p + a_0)} = \frac{W_{st}(p)}{p^{\nu}},$$

then the hodograph of the open system must be supplemented by an arc of a circle of infinitely large radius, which will have ν quarters, i.e. $\nu\frac{\pi}{2}$. A circle of infinite radius is always constructed in the negative direction (clockwise), and if the boundary

$\lim\limits_{p\to 0} W_{st}(p) = \frac{b_0}{a_0} > 0$, the arc of a circle of infinite radius begins on the positive part of the weight axis, and if the boundary $\lim\limits_{p\to 0} W_{st}(p) = \frac{b_0}{a_0} < 0$, then on the negative. The function $W_{st}(p)$ is separated from the transfer function of the open circuit $W_{open}(p)$ by exclusion from the denominator p^{v}. The arc of a circle of infinite radius should end on the hodograph of the open system (Fig. 23). In the case of an astatic open system, the number of negative transitions must include transitions introduced by an arc of infinitely large radius at ω =0.

Example. Investigate the stability of a closed system using the Nyquist criterion (see Fig. 17). Transfer functions:

$$\mathrm{W(p)} = \frac{\mathrm{p}+1}{\mathrm{p}-2}; \qquad \mathrm{W_0(p)} = \frac{10}{\mathrm{p}}.$$

Let's define the transfer function of the open circuit of the system

$$W_{open} = W(p)W_0(p) = \frac{10(p+1)}{p(p-2)}.$$

In the open state, the system is unstable (has a positive root p = 2) and has first-order astatism (one zero root). In the expression $W_{open}(p)$ instead of p we substitute jω we find the Nyquist plot $W_{open}(j\omega)$ of the open system

$$W_{open}(j\omega) = \frac{10(j\omega+1)(-j\omega-2)}{j\omega(j\omega-2)(-j\omega-2)} = 10\frac{\omega^2-2-3j\omega}{j\omega(\omega^2+4)}.$$

From here we get

$$\mathrm{U}(\omega) = -\frac{30}{\omega^2+4}; \qquad \mathrm{V}(\omega) = -10\frac{\omega^2-2}{\omega(\omega^2+4)}.$$

Define the boundary points of the characteristic

$$U(0) = \lim_{\omega\to 0}\left(-\frac{30}{\omega^2+4}\right) = -7.5; \quad V(0) = \lim_{\omega\to 0}\left(-10\frac{\omega^2-2}{\omega(\omega^2+4)}\right) = \infty;$$

$$U(\infty) = \lim_{\omega\to \infty}\left(-\frac{30}{\omega^2+4}\right) = 0\ ; \qquad V(\infty) = \lim_{\omega\to \infty}\left(-10\frac{\omega^2-2}{\omega(\omega^2+4)}\right) = 0.$$

From the obtained expressions for V(ω) and U(ω) it follows that when ω changes from zero to infinity U(ω) is always negative and rotates to zero only when $\omega \to \infty$, and V(ω) takes positive and negative values, with this when $\omega = \sqrt{2}\ \ V(\omega) = 0$ and there is a point of intersection of the abscissa axis.

Substituting $\omega = \sqrt{2}\quad U(\omega) = 0$, we get

$$U(\sqrt{2}) = -\frac{30}{2+4} = -5.$$

Thus, the characteristic $W_{open}(j\omega)$ intersects the abscissa axis at the point (-5, j0).

Since the open circle of the system has a first-order astatism, the hodograph $W_{open}(j\omega)$ must be supplemented at ω = 0 by an arc of a quarter of a circle of infinite radius. Let's define the limit for a static transfer function

$$\lim_{p\to 0} W_{st}(p) = \lim_{p\to 0} \frac{10(p+1)}{p-2} = -5 < 0.$$

Thus, the arc of a circle of infinite radius begins on the negative axis and is a quarter of a circle.

By changing the values of ω from zero to infinity, we construct the Nyquist plot $W_{open}(j\omega)$ of the open part of the system (Fig. 24). To the left of the point (-1, j0) there are two transitions of the characteristic through the weight axis. The point ω = 0 corresponds to half of the negative transition, and the point $\omega = \sqrt{2}$ – a positive transition. The difference between the number of positive and negative transitions is ½. Since in the open state the system was unstable and had one positive root (instability index L = 1), then, according to the Nyquist criterion, we can conclude that in the closed state the system will be stable, as the difference between the number of positive and negative transitions L/2 = ½.

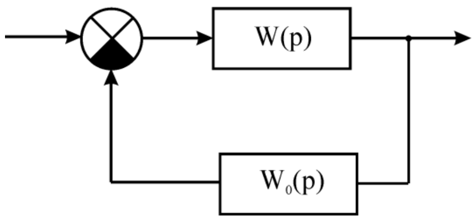

Fig. 17

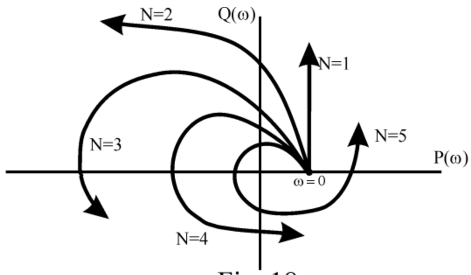

Fig. 18

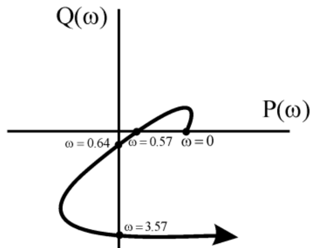

Fig. 19

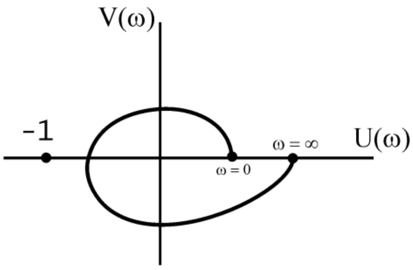

Fig. 20

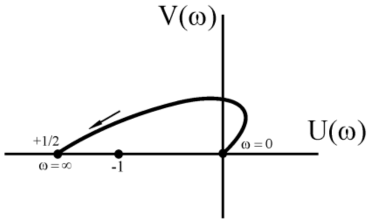

Fig. 21

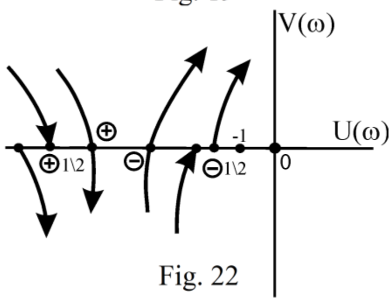

Fig. 22

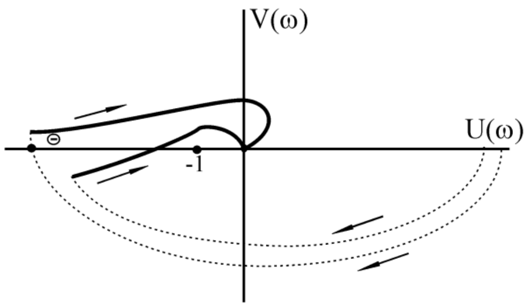

Fig. 23

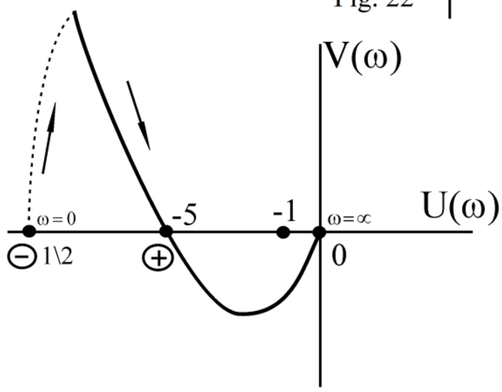

Fig. 24

<u>Task 4.</u> Solve the linear difference equation using a discrete Laplace transform. The initial data are given in table 6. Initial conditions are zero.

Guidelines

1. Get acquainted with the method of compiling and solving linear difference equations [1, 2, 11].

2. To study the discrete Laplace transform and z-transform, decomposition formulas [1, 2, 11].

Table 6

Variant number	a_3	a_2	a_1	a_0	g(mT)
1	0	2	K	1	5mT
2	K	2	1	0	10
3	0	1	2	1	4K
4	0	1	0	1	2K
5	1	4	5K	0	5
6	0	4	K	6	4mT
7	0	0	K	2	3+2mT
8	1	K	2	0	10mT
9	0	K	4K	4K	5K
10	1	2K	10	0	12mT
11	1	K	20	0	e^{mT}
12	0	10	K	1	e^{2mT}
$g(mT) = a_3 x((m+3)T) + a_2 x((m+2)T) + a_1 x((m+1)T) + a_0 x(mT)$					

The discrete Laplace transform matches the lattice function-original $x(mT)$ ($x(mT) \equiv 0$; m <0; m – period number, m = 0,1,2…; T – quantization period) with the image X*(p). Conditionally, such a transition is written as follows:

$$X^*(p) = D\{x(mT)\}.$$

The following properties of the discrete Laplace transform are used when solving linear difference equations

1. The property of linearity

$$D\left\{\sum_{k=1}^{n} a_k x_k(mT)\right\} = \sum_{k=1}^{n} a_k D\{x_k(mT)\} = \sum_{k=1}^{n} a_k X_p^*(p).$$

2. Displacement of the argument of the lattice function

$$D\{x(m-s)T\} = e^{-psT} D\{x(mT)\} = e^{-psT} X^*(p);$$

$$D\{x(m-s)T\} = e^{-psT}\left[X^*(p) - \sum_{k=0}^{s-1} x(kT) e^{-psT}\right],$$

where X(кT) – initial values of the lattice function at points *k*. At zero initial conditions $x(0) = x(T) = \ldots x((s-1)T) = 0;\ D\{x((m+s)T)\} = e^{psT} X^*(p).$

Images of the most common functions.

1. Unit stepped function $g(mT)=1;\ D\{1\}=\dfrac{e^{pT}}{(e^{pT}-1)}$.

2. Linear function $g(mT)=mT;\ D\{mT\}=\dfrac{Te^{pT}}{(e^{pT}-1)^2}$.

3. Exponential function $g(mT)=e^{\alpha mT};\ D\{e^{\alpha mT}\}=\dfrac{e^{pT}}{(e^{pT}-e^{\alpha T})}$.

The inverse discrete Laplace transform allows us to determine the lattice function from its image X*(p)

$$x(mT)=D^{-1}\left\{X^*(p)\right\}=\sum_{k=1}^{n}\operatorname{Re}s\left[TX^*(p)e^{pmT}\right]_{p=p_k},$$

where n – is the number of poles of the function X*(p) (zeros of the characteristic equation); p_k – is the *k*-th pole of the function X*(p).

If all the poles of the function X*(p) are different, then the lattice function can be found by the formula

$$x(mT)=\sum_{k=1}^{n}T\frac{B^*(p_k)}{A^{*`}(p_k)}e^{p_k mT}l;$$

$$X^*(p)=\frac{B^*(p)}{A^*(p)}\ ;\quad A^{*\prime}(p_k)=\left.\frac{dA^*(p)}{dp}\right|_{p=p_k}.$$

Usually, when performing a reverse transition, pre-replace e^{pT} with z.

Then for the case of simple poles

$$x(mT)=\sum_{k=1}^{n}\operatorname{Re}s\left[X^*(z)z^{m-1}\right]_{z=z_k}=\sum_{k=1}^{n}\frac{B^*(z_k)}{A^{*\prime}(z_k)}z_k^{\,m-1},\quad m\geq 1,$$

and for the case of multiple poles

$$x(mT)=\sum_{k=1}^{n}\frac{1}{(S_k-1)!}\frac{d^{S_k-1}}{dz^{S_k-1}}\left[(z-z_k)^{S_k}\frac{B^*(z)}{A^*(z)}z^{m-1}\right]_{z=z_k},\quad m\geq 1,$$

where S_k – is the multiplicity of the k-th pole; $B^*(z)=b_r z^r+b_{r-1}z^{r-1}+...b_1 z+b_0$; $A^*(z)=a_r z^r+a_{r-1}z^{r-1}+...a_1 z+a_0$.

Function images are often obtained in the form

$$X^*(z) = \frac{z^l}{z^\nu}\frac{B^*(z)}{A^*(z)}.$$

When finding the lattice function of the original in this case

$$x(mT) = \sum_{k=1}^{n} \frac{B^*(z_k)}{A^{*\prime}(z_k)} z_k^{m+l-\nu-1},$$

And m can take values

$$m \ge \begin{cases} \nu + 1 - l, \nu + 1 - l > 0; \\ 0 \qquad , \nu + 1 - l < 0. \end{cases}$$

Example. Solve the difference equation under zero initial conditions

$$a_2 x((m+2)T) + a_1 x((m+1)T) + a_0 x(mT) = 1, \text{ где } a_2 = 1; a_1 = 5; a_0 = 6.$$

Apply to the left and right parts of the difference equation the D-transform:

$$D\{x((m+2)T) + 5x((m+1)T) + 6x(mT)\} = D\{1\}.$$

In accordance with the properties of linearity and displacement

$$e^{2pT}X^*(p) + 5e^{pT}X^*(p) + 6X^*(p) = \frac{e^{pT}}{e^{pT} - 1}.$$

Solve this equation accordingly

$$X^*(p) = \frac{e^{pT}}{(e^{pT} - 1)(e^{2pT} + 5e^{pT} + 6)}.$$

We make a replacement

$$e^{pT} = z.$$

To perform the inverse transition, find the roots of the characteristic equation $(z-1)(z^2 + z + 6) = 0$ and determine $z_1 = 1; z_2 = -2; z_3 = -3.$

Let's use the inversion formula for the case of simple roots:

$$x(mT) = \sum_{k=1}^{3} \frac{B^*(z_k)}{A^{*\prime}(z_k)} z_k^{m-1} = \sum_{k=1}^{3} \frac{z_k}{A^{*\prime}(z_k)} z_k^{m-1} = \sum_{k=1}^{3} \frac{1}{A^{*\prime}(z_k)} z_k^{m},$$

where $x(mT) = \frac{1}{12} - \frac{1}{3}(-2)^m + \frac{1}{4}(-3)^m;\ A^{*\prime}(z_k) = 3z_k^2 + 8z_k + 1.$

<u>Task 5.</u> Find the transfer function of pulse automatic control system. The initial data are given in table 7. The schemes are shown in Fig. 25-32.

Guidelines

1. Get acquainted with the $\overline{D}$-transform, which establishes a connection between continuous and discrete Laplace transforms, as well as with the properties of the $\overline{D}$-transform [7, 8].

Table 7

Variant number	Figure number	$W_1(p)$	$W_2(p)$
1	25	10K/[(p+1)(p+10)]	-
2	26	100K/(p+2)	10K/(p+10)
3	27	K/(p+10)	20/(p-20)
4	28	20/(p+10)	15K/(p-20)
5	29	100/p	10K/(p-10)
6	30	10/p	1/(p+K)
7	31	10/(p+10)	10K/(p+20)
8	32	K/(p+20)	(p+1)/(p+5)
9	25	100K/[(p-10)(p-20)]	-
10	26	10/(p+10)	pK/(p+1)
11	27	p/(p+0.5)	K/(p+40)
12	28	20p/(p+10)	0.1/(p+K)

2. To study the method of compiling the equations of pulse automatic control system and find the transfer functions of open and closed systems.

If the image of a continuous function X(p) is known, then the image of the lattice function is determined by means of $\overline{D}$-transform.

$$X^*(p) = \overline{D}\{X(p)\} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(p - jk\omega) + \frac{x(0)}{2},$$

where $\omega = \frac{2\pi}{T}$; $x(0) = x(t)\big|_{t=0}$.

If the poles of the function X(p) are known, then the image of the lattice function

$$X^*(p) = \sum_{k=1}^{n} \mathrm{Res}\left[X(q)\frac{e^{pT}}{e^{pT} - e^{qT}}\right]_{q=p_k},$$

where p_k – is the k-th pole of the function X(p).

For the case of different poles of the function X(p)

$$X^*(p)=\sum_{k=1}^{n}\frac{B(p_k)}{A`(p_k)}\frac{e^{pT}}{e^{pT}-e^{p_kT}}.$$

To find the transfer function of pulse systems, we use the following properties of the $\overline{D}$-transform.

1. Multiplication of images of continuous and lattice functions

$$\overline{D}\{X_1(p)X_2^*(p)\}=\overline{D}\{X_1(p)\}X_2^*(p)=X_1^*(p)X_2^*(p).$$

If $X_1(p)=e^{pT}$, then $\overline{D}\{e^{pT}X_2^*(p)\}=e^{pT}X_2^*(p).$

2. Linearity of the $\overline{D}$-transform

$$\overline{D}\left\{\sum_{k=1}^{n}a_kX_k(p)\right\}=\sum_{k=1}^{n}a_k\overline{D}\{X_k(p)\}=\sum_{k=1}^{n}a_kX_k^*(p).$$

For an open pulse system (Fig. 33), the transfer function can be found using $\overline{D}$-transform

$$W^*(p)=\overline{D}\{W(p)\}=\overline{D}\{S(p)W_н(p)\}.$$

For a closed pulse system (Fig. 34) the equations have the form

$$X(p)=S(p)W_н(p)\xi^*(p)=\mathrm{W(p)}\xi^*(p)\ ;\qquad \xi(p)=G(p)-W_0(p)X(p).$$

Substituting the first equation into the second, we obtain

$$\xi(p)=G(p)-W(p)W_0(p)\xi^*(p).$$

Using the properties of linearity and multiplication of images of continuous and lattice functions of $\overline{D}$-transform, we obtain

$$\overline{D}\{\ \xi(p)\}=\overline{D}\{G(p)-\overline{D}\{W(p)W_0(p)\xi^*(p)\}\}\ ;$$
$$\xi^*(p)=G^*(p)-\overline{D}\{W(p)W_0(p)\}\xi^*(p).$$

Solve the resulting equation with respect to

$$\xi^*(p)=\frac{G^*(p)}{1+\overline{D}\{W(p)W_0(p)\}}.$$

Apply $\overline{D}$-transform to the first equation of the system:

$$\overline{D}\{\ X(p)\}=\overline{D}\{W(p)\xi^*(p)\};$$
$$X^*(p)=\overline{D}\{W(p)\}\xi^*(p).$$

Substituting in this equation the received expression for $\xi^*(p)$, we will write down

$$X^*(p) = \overline{D}\{W(p)\}\frac{G^*(p)}{1+\overline{D}\{W(p)W_0(p)\}}.$$

Then for the transfer function of the pulse automatic control system we obtain

$$W_з^*(p) = \frac{X^*(p)}{G^*(p)} = \frac{\overline{D}\{W(p)\}}{1+\overline{D}\{W(p)W_0(p)\}}.$$

Example. Find the transfer function of a closed pulse system shown in Fig. 35. Transfer functions of links

$$S(p) = \frac{1-e^{-pT}}{p};\ W_н(p) = \frac{10}{p+5};\ W_2(p) = \frac{1}{p+20}.$$

Define the transfer function of the link covered by a single feedback

$$W_1(p) = \frac{X(p)}{Y(p)} = \frac{W_н(p)}{1+W_н(p)} = \frac{10}{p+15}.$$

Write the equation for the pulse system:

$$X(p) = S(p)W_1(p)\xi^*(p);$$
$$\xi(p) = G(p) - W_2(p)X(p).$$

Substituting the first equation into the second, we obtain

$$\xi(p) = G(p) - W_2(p)S(p)W_1(p)\xi^*(p).$$

Apply $\overline{D}$-transform:

$$\xi^*(p) = G^*(p) - \overline{D}\{W_2(p)S(p)W_1(p)\}\xi^*(p)$$

or

$$\xi^*(p) = \frac{G^*(p)}{1+\overline{D}\{W_2(p)S(p)W_1(p)\}}.$$

Applying $\overline{D}$-transform to the first equation of the system, we obtain $X^*(p) = \overline{D}\{S(p)W_1(p)\}\xi^*(p).$

Taking into account the last two expressions, the transfer function of a closed pulse system will have the following form:

$$W_з^*(p) = \frac{X^*(p)}{G^*(p)} = \frac{\overline{D}\{S(p)W_1(p)\}}{1+\overline{D}\{W_2(p)S(p)W_1(p)\}}.$$

Find $\overline{D}\{S(p)W_1(p)\}$ with the poles of the continuous image

$$\overline{D}\{S(p)W_1(p)\} = \overline{D}\left\{\frac{1-e^{-pT}}{p}\frac{10}{p+15}\right\}.$$

Let's use the properties of linearity and multiplication of the image on the exponent:

$$\overline{D}\left\{\frac{1-e^{-pT}}{p}\frac{10}{p+15}\right\} = 10\overline{D}\left\{\frac{1}{p(p+15)}\right\} - \overline{D}\left\{\frac{10e^{-pT}}{p(p+15)}\right\} =$$
$$= 10\overline{D}\left\{\frac{1}{p(p+15)}\right\} - 10e^{-pT}\overline{D}\left\{\frac{1}{p(p+15)}\right\} = 10(1-e^{-pT})\overline{D}\left\{\frac{1}{p(p+15)}\right\}.$$

Thus, the constant factor and the expression of the species $1-e^{-pT}$ can be taken out for a sign of $\overline{D}$-transfor.

Because the poles of the function $\frac{1}{p(p+15)}$ $\quad p_1 = 0,\ p_2 = -15,$ then

$$\overline{D}\left\{\frac{1}{p(p+15)}\right\} = \sum_{k=1}^{2}\frac{B(p_k)}{A`(p_k)}\frac{e^{pT}}{e^{pT}-e^{p_kT}} = \sum_{k=1}^{2}\frac{1}{2p_k+15}\frac{e^{pT}}{e^{pT}-e^{p_kT}} =$$
$$= \frac{1}{15}\frac{e^{pT}}{e^{pT}-e^{p_kT}} - \frac{1}{15}\frac{e^{pT}}{e^{pT}-e^{-15T}} = \frac{1}{15}\frac{e^{pT}(1-e^{-15T})}{(e^{pT}-e^{p_kT})(e^{pT}-e^{-15T})}.$$

And

$$10(1-e^{-pT})\overline{D}\left\{\frac{1}{p(p+15)}\right\}=\frac{10(e^{pT}-1)}{e^{pT}}\frac{1}{15}\frac{e^{pT}(1-e^{-15T})}{(e^{pT}-e^{p_kT})(e^{pT}-e^{-15T})}=\frac{2}{3}\frac{(1-e^{-15T})}{(e^{pT}-e^{-15T})}.$$

Similarly for $\overline{D}\{W_2(p)S(p)W_1(p)\}$ we get

$$\overline{D}\left\{\frac{1-e^{-pT}}{p}\frac{10}{p+15}\frac{1}{p+20}\right\}=10(1-e^{-pT})\overline{D}\left\{\frac{1}{p(p+15)(p+20)}\right\}.$$

Poles of the function $p_1=0,\ p_2=-15,\ p_3=-20,$

$$\overline{D}\left\{\frac{1}{p(p+15)(p+20)}\right\}=\sum_{k=1}^{3}\frac{1}{3p_k^2+7p_k+300}\frac{e^{pT}}{e^{pT}-e^{p_kT}}=$$
$$=\frac{1}{300}\frac{e^{pT}}{e^{pT}-1}-\frac{1}{15}\frac{e^{pT}}{e^{pT}-e^{15T}}+\frac{1}{100}\frac{e^{pT}}{e^{pT}-e^{20T}}=$$
$$=\frac{e^{pT}}{300}\left[\frac{e^{2pT}-(e^{-15T}+e^{-20T})e^{pT}+e^{-35T}-4e^{2pT}+4(e^{-20T}+1)e^{pT}-4e^{-20T}+3e^{2pT}-3(1+e^{-15T})e^{pT}+3e^{-15T}}{(e^{pT}-1)(e^{pT}-e^{-15T})(e^{pT}-e^{-20T})}\right]=$$
$$=\frac{e^{pT}}{300}\left[\frac{e^{pT}(1+3e^{-20T}-4e^{-15T})+e^{-35T}-4e^{-20T}+3e^{-15T}}{(e^{pT}-1)(e^{pT}-e^{-15T})(e^{pT}-e^{-20T})}\right].$$

Then

$$10(1-e^{-pT})\overline{D}\left\{\frac{1}{p(p+15)(p+20)}\right\}=\frac{1}{30}\left[\frac{e^{pT}(1+3e^{-20T}-4e^{-15T})+e^{-35T}-4e^{-20T}+3e^{-15T}}{(e^{pT}-e^{-15T})(e^{pT}-e^{-20T})}\right].$$

Taking into account the found expressions, the transfer function of a closed system will be written as follows

$$W^*_{closed}(p)=\frac{2\left(e^{pT}-e^{-20T}\right)\left(1-e^{-15T}\right)}{90e^{2pT}+3\left(1-27e^{-20T}-34e^{-15T}\right)e^{pT}+93e^{-35T}-12e^{-20T}+9e^{-15T}}.$$

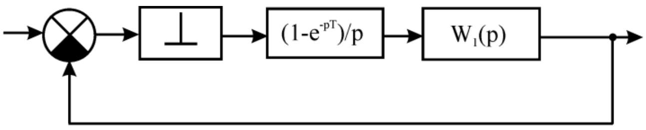

Fig. 25

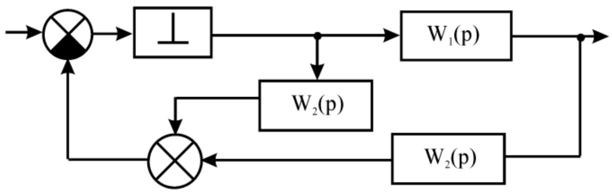

Fig. 26

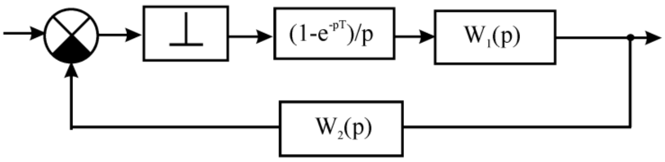

Fig. 27

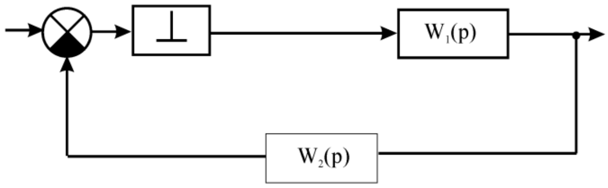

Fig. 28

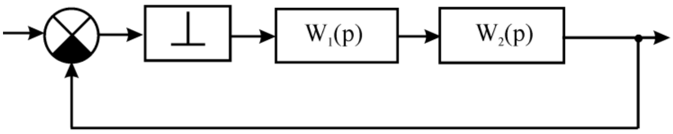

Fig. 29

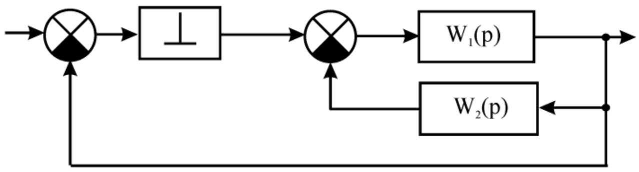

Fig. 30

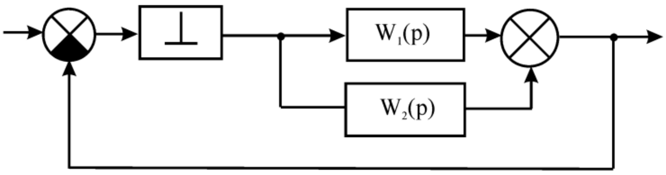

Fig. 31

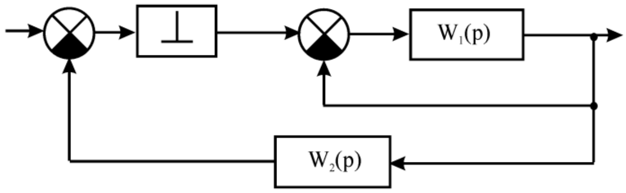

Fig. 32

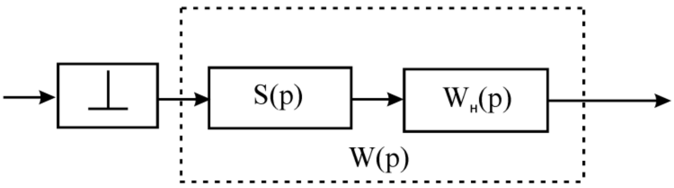

Fig. 33

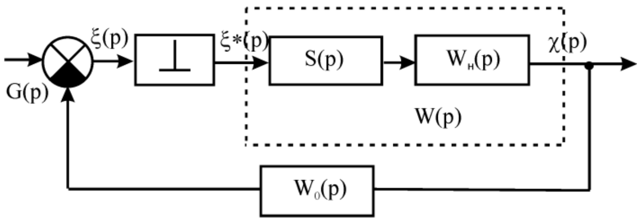

Fig. 34

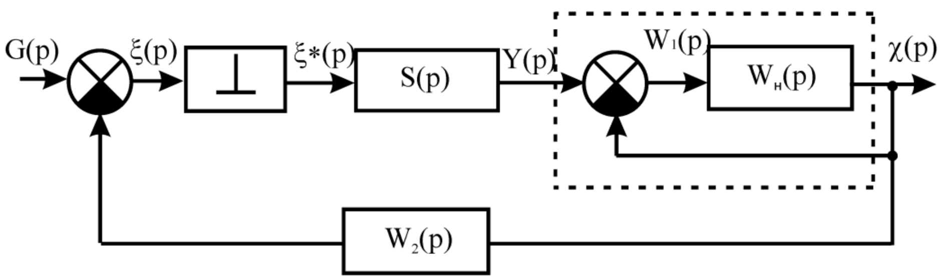

Fig. 35

<u>Task 6</u>. Explore the stability of the closed system by the following methods:

1) analogue of the Hurwitz criterion, using $W^*_{closed}(p)$, obtained in task 5 (quantization period T = 0,1);

2) analogue of the Routh criterion (initial data are given in table 8);

3) analogue of the Mikhailov criterion (initial data are given in table 8), scheme of the pulse system is shown in Fig. 25, $W_1(p) = \dfrac{K_1}{(p+a)(p+b)}$.

Guidelines

1. Get acquainted with the concept of "stability" for linear pulsed systems and with the necessary and sufficient condition of stability [2, 4, 9].

2. To study algebraic criteria of stability (analogues of Hurwitz and Routh criteria) [9].

3. To study the frequency criteria of stability (analogues of Mikhailov and Nyquist criteria) [9].

Table 8

Variant number	**Task 6, 2**					**Task 6, 3**			
	a_0	**a_1**	**a_2**	**a_3**	**a_4**	**K_1**	**A**	**b**	**T**
1	K	10	1.1K	50	1.5K	100	1	3	0.1
2	2K	7	K	15	10K	10K	K	2K	0.01
3	3	2K	30	K	20	20	0.1K	0.3K	0.1
4	45	20	K+5	12	80	5K	3	7	0.1
5	10	1.5K	20	K+10	45	50	0.2K	20	0.1
6	5	K+20	0.1K	40	10	30	15	0.1K	0.1
7	0.3K	13	5K	18	5	70	35	K	0.01
8	5.5	10	0.2K	0.5K	20	2K	15	30	0.1
9	0.1K	2.5	4.2	0.7K	50	100	3K	5K	0.01
10	0.5	1.8	2.2	0.1K	3	4K	0.1K	0.3K	0.1
11	10	2K	K+3	17	25	2.5K	5	2	0.1
12	K+5	15	20	35	2K	45	8	0.1K	0.1
$A^*(p) = a_4 e^{4pT} + a_3 e^{3pT} + a_2 e^{2pT} + a_1 e^{pT} + a_0$									

For an open pulsed system, the stability can be investigated by the transfer function of its continuous part. If the continuous part is stable, neutral or unstable, then the open pulsed system is stable, neutral or unstable, respectively. Thus, analogues of the criteria of Hurwitz, Routh, Mikhailov and others is expedient to use only for research of steady closed pulsed systems. When using the listed criteria it is necessary to find preliminary transfer function of the closed pulsed system

$$W^{*}_{closed}(p)=\frac{b_m e^{pmT}+b_{m-1}e^{p(m-1)T}+....+b_1 e^{pT}+b_0}{a_n e^{pnT}+a_{n-1}e^{p(n-1)T}+....+a_1 e^{pT}+a_0}.$$

A characteristic polynomial (denominator of the transfer function) is used to study the stability

$$A^{*}(p)=a_n e^{pnT}+a_{n-1}e^{p(n-1)T}+....+a_1 e^{pT}+a_0.$$

Analogue of the Hurwitz criterion. Let's replace variables $e^{pT}=z$:

$$A^{*}(z)=a_n z^{n}+a_{n-1}z^{n-1}+....+a_1 z+a_0.$$

In order to be able to use the Hurwitz criterion in its usual formulation, it is necessary to make another substitution of variables

$$z=\frac{u+1}{u-1}.$$

Then we get

$$A^{*}\left(\frac{u+1}{u-1}\right)=a_n\left(\frac{u+1}{u-1}\right)^{n}+a_{n-1}\left(\frac{u+1}{u-1}\right)^{n-1}+....+a_1\left(\frac{u+1}{u-1}\right)+a_0$$

or

$$(u-1)^{n}A^{*}\left(\frac{u+1}{u-1}\right)=A^{*}(u)=a_n(u+1)^{n}+a_{n-1}(u+1)^{n-1}(u-1)+....+a_1(u+1)(u-1)^{n-1}+a_0(u-1)^{n}.$$

After opening brackets and reduction of similar terms we will receive

$$A^{*}(z)=A_n u^{n}+A_{n-1}u^{n-1}+....+A_1 u+A_0.$$

The impulse system will be stable if and only if *n* main determinants of the following matrix of coefficients of the characteristic equation of the system are positive:

$$\Delta n=\begin{vmatrix} A_{n-1} & A_{n-3} & A_{n-5} & A_{n-7} & ...... & 0 \\ A_n & A_{n-2} & A_{n-4} & A_{n-6} & .... & 0 \\ 0 & A_{n-1} & A_{n-3} & A_{n-5} & .... & 0 \\ 0 & A_n & A_{n-2} & A_{n-4} & .... & 0 \\ .... & .... & .... & .... & .... & ...... \\ 0 & 0 & 0 & .... & A_1 & 0 \\ 0 & 0 & 0 & .... & A_2 & A_0 \end{vmatrix}$$

that is $A_n>0; \Delta_1=A_{n-1}>0....\Delta_n>0.$

Example. Investigate the stability of a closed pulsed system (see Fig. 25) using analogue of the Hurwitz criterion

$$W_1(p)=\frac{10}{(p+2)(p+5)},\ T=0.1.$$

Let's find the transfer function of a closed pulsed system

$$W^*_{closed}(p)=\frac{\bar{D}\left\{\dfrac{1-e^{-pT}}{p}W_1(p)\right\}}{1+\bar{D}\left\{\dfrac{1-e^{-pT}}{p}W_1(p)\right\}}$$

and function

$$\bar{D}\left\{\frac{1-e^{-pT}}{p}W_1(p)\right\}=\bar{D}\left\{\frac{1-e^{-pT}}{p}\frac{10}{(p+2)(p+5)}\right\}=10(1-e^{-pT})\bar{D}\left\{\frac{1}{p(p+2)(p+5)}\right\}.$$

Poles of continuous function $p_1=0,\ p_2=-2, p_3=-5,$ then

$$\bar{D}\left\{\frac{1}{p(p+2)(p+5)}\right\}=\sum_{k=1}^{3}\frac{1}{3p_k^2+14p_k+10}\frac{e^{pT}}{e^{pT}-e^{p_kT}}=$$
$$=\frac{1}{10}\frac{e^{pT}}{e^{pT}-1}-\frac{1}{6}\frac{e^{pT}}{e^{pT}-e^{-2T}}+\frac{1}{15}\frac{e^{pT}}{e^{pT}-e^{-5T}}=$$
$$=\frac{e^{pT}}{30}\left[\frac{e^{pT}(3+2e^{-5T}-5e^{-2T})+3e^{-7T}-5e^{-5T}+2e^{-2T}}{(e^{pT}-1)(e^{pT}-e^{-2T})(e^{pT}-e^{-5T})}\right];$$

$$10(1-e^{-pT})\bar{D}\left\{\frac{1}{p(p+2)(p+5)}\right\}=\frac{1}{3}\left[\frac{e^{pT}(3+2e^{-5T}-5e^{-2T})+3e^{-7T}-5e^{-5T}+2e^{-T}}{(e^{pT}-1)(e^{pT}-e^{-2T})(e^{pT}-e^{-5T})}\right].$$

Taking into account the obtained expression, the transfer function of a closed pulsed system

$$W^*_{closed}(p)=\frac{e^{pT}(3+2e^{-5T}-5e^{-2T})+3e^{-7T}-5e^{-5T}+2e^{-T}}{3e^{2pT}+(3-e^{-5T}-8e^{-2T})e^{pT}+6e^{-7T}-5e^{-5T}+2e^{-2T}}.$$

Substituting 0,1 instead of T, we obtain:

$$W^*_{closed}(p)=\frac{0{,}119e^{pT}+0{,}095}{3e^{2pT}-4{,}156e^{pT}+1{,}584}=\frac{0{,}039e^{pT}+0{,}0317}{e^{2pT}-1{,}385e^{pT}+0{,}528}.$$

Characteristic polynomial $A^*(p)=e^{2pT}-1.385e^{pT}+0.528.$

Replacing $e^{pT}=z$, we get $A^*(z)=z^2-1.385z+0.528.$

Substituting $z=\dfrac{u+1}{u-1}$, we get

$$A^*(u) = (u+1)^2 - 1.385(u+1)(u-1) + 0.528(u-1)^2.$$

Opening brackets, we reduce similar terms $A^*(u) = 0.143u^2 + 0.944u + 2.913.$

We make Hurwitz's determinant:

$$\Delta_2 = \begin{vmatrix} 0.944 & 0 \\ 0.143 & 2.913 \end{vmatrix}.$$

Since $A_2 = 0.143 > 0; \Delta_1 = 0.944 > 0;\ \Delta_2 = 2.75 > 0,$ then according to Hurwitz criterion, we can conclude that this pulsed system is stable.

Analogue of the Routh criterion. This criterion allows us to study the stability of the pulse system directly by the coefficients of the characteristic polynomial $A^*(p) = a_n e^{pnT} + a_{n-1}e^{p(n-1)T} + ... + a_1 e^{pT} + a_0$. Let's make an analogue of the Routh table according to the following rule (table 9). In the first pair of rows of the table, we enter the coefficients of the polynomial $A^*(p)$ in ascending order, and below them – in reverse order. The coefficients of the following pairs of rows are determined in the following way. From the coefficients of the upper row of each pair subtract the coefficients of the lower row, previously multiplied by such a number that the first difference turns into zero. Rejecting this zero difference, we find the first line of the pair. The second line consists of the same coefficients, but written in reverse order.

Table 9

Coefficient	a_0 a_n	a_1 a_{n-1}	………..	a_{n-1} a_1	a_n a_0
$\lambda_1 = \frac{a_0}{a_n} < 1$	$c_{0,2} = a_1 - \lambda_1 a_{n-1}$ $c_{n-1,2}$	$c_{1,2} = a_2 - \lambda_1 a_{n-2}$ $c_{n-2,2}$	………….	$c_{n-1,2} = a_n - \lambda_1 a_0$ $c_{0,2}$	………….
$\lambda_2 = \frac{c_{0,2}}{c_{n-1,2}}$	$c_{0,3} = c_{1,2} - \lambda_2 c_{n-2,2}$ $c_{n-2,3}$	$c_{1,3} = c_{2,2} - \lambda_2 c_{n-3,2}$ $c_{n-3,3}$	………….	………….	………….
…………….	……………….	……………….	………….	………….	………….

Any of the coefficients $C_{k,i}$ (i>2) of the table, where the first index means the column number (coefficient number), and the second – the row number in which the coefficient is located, can be found by the formula

$$c_{k,i} = c_{k+1,i-1} - \lambda_{i-1} c_{n-i-k+1,i-1},$$

where $\lambda_i = \dfrac{c_{0,i}}{c_{n-i+1,i}}$.

Using the data in this table and Schur's theorem, the criterion of stability can be formulated as follows: in order for a closed system to be stable, it is necessary and sufficient that the coefficients λ_i are less than one in absolute value, i.e. $|\lambda_i| < 1$.

Example. Investigate the stability of a closed pulse system using an analogue of the Routh criterion (the initial data are the same as in the previous example).

Characteristic polynomial $A^*(p) = e^{2pT} - 1.385e^{pT} + 0.528$.

Let's make an analogue of the Routh table (table 10).

Table 10

Coefficient	$a_0 = 0.528$ $a_2 = 1$	$a_1 = -1.385$ $a_1 = -1.385$	$a_2 = 1$ $a_0 = 0.528$
$\lambda_1 = \dfrac{0.528}{1} = 0.528$	$c_{0,2} = -1.385 - 0.528(-1.385) =$ $= -0.954$ $c_{1,2} = 0.721$	$c_{1,2} = 1 - 0.528 \cdot 0.528 = 0.721$ $c_{0,2} = -0.654$	0 0
$\lambda_2 = \dfrac{-0.654}{0.721} = -0.907$	$c_{0,3} = 0.721 - (-0.907)(-0.654) =$ $= 0.128$ 0	0 0	0 0

Since $|\lambda_1| < 1$ and $|\lambda_2| < 1$, we conclude that this closed system is stable.

Analogue of Mikhailov criterion. For the stability of a linear pulsed system, it is necessary and sufficient that when the frequency ω changes from 0 to π/T, the vector $A^*(j\omega)$ passes counterclockwise 2*n* quadrants, where *n* is the degree of the characteristic polynomial.

To build the Mikhailov curve (hodograph), p should be replaced by jω in $A^*(j\omega)$, and then the real and imaginary parts of $A^*(j\omega) = P^*(j\omega) + jQ^*(j\omega)$ can be distinguished. In the plane of the parameters $P^*(\omega)$ and $Q^*(\omega)$ we build a hodograph, the type of which can be used to judge the stability of the system. Fig. 36 shows a hodograph of the stable system, and Fig. 37 – hodograph of the unstable system (n = 3).

<u>Example.</u> Investigate the stability of the pulse system (Fig. 38) using the Mikhailov criterion.

Find the transfer function of a closed pulsed system

$$W^*_{closed}(p) = \frac{\bar{D}\left\{\dfrac{1-e^{-pT}}{p}\dfrac{K_0}{T_1 p + 1}\right\}}{1 + \bar{D}\left\{\dfrac{1-e^{-pT}}{p}\dfrac{K_0}{T_1 p + 1}\right\}}.$$

Using the properties of $\bar{D}$-transform, we obtain:

$$\bar{D}\left\{\frac{1-e^{-pT}}{p}\frac{K_0}{T_1 p+1}\right\} = K_0(1-e^{-pT})\bar{D}\left\{\frac{1}{p(T_1 p+1)}\right\} =$$

$$= K_0(1-e^{-pT})\sum_{k=1}^{n}\frac{B(p_k)}{A^*(p_k)}\frac{e^{pT}}{e^{pT}-e^{p_k T}}\left|\begin{matrix} p_1 = 0 \\ p_2 = -\dfrac{1}{T_1}\end{matrix}\right| =$$

$$= K_0(1-e^{-pT})\sum_{k=1}^{2}\frac{1}{2T_1 p_k + 1}\frac{e^{pT}}{e^{pT}-e^{p_k T}} = K_0(1-e^{-pT})\left[\frac{e^{pT}}{e^{pT}-1} - \frac{e^{pT}}{e^{pT}-e^{-T/T_1}}\right] =$$

$$= K_0\frac{1-e^{-T/T_1}}{e^{pT}-e^{-T/T_1}}.$$

Transfer function of a closed system

$$W^*_{closed}(p) = \frac{K_0\left(1-e^{-T/T_1}\right)}{e^{pT} - e^{-T/T_1} + K_0\left(1-e^{-T/T_1}\right)}$$

Characteristic polynomial

$$A^*(p) = e^{pT} - e^{-T/T_1} + K_0(1-e^{-T/T_1}).$$

Let's select the real and imaginary parts

$$A^*(j\omega) = e^{j\omega T} - e^{-T/T_1} + K_0(1-e^{-T/T_1}) = \cos\omega T + j\sin\omega T - e^{-T/T_1} + K_0(1-e^{-T/T_1});$$

$$P^*(\omega) = \cos\omega T + -e^{-T/T_1} + K_0(1-e^{-T/T_1});$$

$$Q^*(\omega) = \sin\omega T.$$

Fig. 39 shows Mikhailov hodographs for a system with parameters K_0 = 20; T = 0,2 s; T_1 = 1 s (curve 1) and systems with parameters K_0 = 10; T = 0,2 s; T_1 = 1 s (curve 2). Thus, for the system shown in Fig. 38 (*n* = 1), the stability depends on the value of the gain coefficient K_0. At K_0 = 20 the system is unstable (curve 1), and at K_0 = 10 – stable (curve 2).

Task 7. Investigate the stability of the automatic control system using the first Lyapunov method, if the equations of a closed system have the form

$$\frac{dy_i(t)}{dt} = \Phi_i(y_1, y_2, y_3),\ i = \overline{1,3}.$$

The initial data are given in table 11.

Table 11

Variant number	Φ_1 (y_1, y_2, y_3)	Φ_2 (y_1, y_2, y_3)	Φ_3 (y_1, y_2, y_3)
1	$2y_1 - 0{,}3Ky_3 + 5y_1y_2$	$-5y_1 + 0.1Ky_1y_2 + y_3$	$y_1 + 0.2Ky_2^2 - y_1y_3 + y_3$
2	$Ky_1 - 0.5e^{-y_2} + 5y_2y_3$	$\sin y_1 + \cos y_2 + 2Ky_3$	$20\mathrm{tg}y_1 + y_1y_2 + 25y_3$
3	$y_1^2 + 3\arcsin y_2 - K^2y_3$	$2e^{y_1} - 5K\sin y_2 - 15y_3$	$y_1 + 20y_1y_2 + (K-10)y_3$
4	$5y_1 + y_2y_3 - (K+10)y_3$	$0.5y_1 + \mathrm{tg}y_2 - K\cos y_3$	$13e^{y_1} - (K-20)y_2 + 4\mathrm{tg}y_3$
5	$0.5Ky_1 + y_1y_2 - 30y_3$	$y_1y_3 + \arccos y_2 + 20y_3$	$15Ky_1 + (K+10)y_2 - 0.2\mathrm{arctg}y_3$
6	$15y_1 - e^{y_2} + 2\cos y_3$	$y_1^2 - (K+10)y_2 - 0.3y_3$	$\sin y_1 - y_1y_2 + 15Ky_3$
7	$4e^{y_1} - \sin y_2 + 3y_3$	$y_1 + \cos y_2 - Ky_3$	$\mathrm{tg}y_1 + 5y_2 - 25y_3$
8	$12y_1 - Ky_2^2 + (K+15)e^{y_3}$	$13y_1 - \sin y_2 + y_2y_3$	$5\arcsin y_1 - 3y_2^2 + (K-7)y_3$
9	$32y_1 + y_1y_2 - 2Ky_3$	$0.7y_2 - y_2y_3 + 8Ky_3$	$17\arccos y_1 - Ky_1y_2 - 21e^{y_3}$
10	$3Ky_1 - 7y_2 + 10y_3$	$5Ke^{y_1} + \sin y_1 - y_2y_3$	$19y_1^2 - (K-5)y_1y_2 + 6Ky_3$
11	$(K+12)y_1^2 - 5e^{y_2} + 13y_3$	$1.3K\sin y_1 + 2y_2 - y_3$	$(K+1)^2y_1 + 2y_1y_2 - 3y_3$
12	$7y_1 - (K+3)y_2 + y_3$	$e^{y_1} + \mathrm{tg}y_2 - 2.3Ky_3$	$y_1y_2 - 1.8y_2 + (K+11)y_3$

Guidelines

1. Get acquainted with the concept of “nonlinear system”, types of nonlinearities [11].

2. To study the first Lyapunov method (study of stability by linearized equations) [10].

The first Lyapunov method is used to study the free motion of the system with respect to the origin (y_i = 0) at infinitesimal deviations. Consider a nonlinear system described by a system of equations

$$\frac{dy_i(t)}{dt} = \Phi_i(y_1, y_2, ..., y_n), \ i = \overline{1, n}.$$

We linearize the nonlinear functions Φ_i around the point $y_i(t)=0$

$$\frac{dy_i(t)}{dt} = a_{i1}y_1(t) + a_{i2}y_2(t) + ... + a_{in}y_n(t) + R_i(y_1, y_2, ..., y_n),$$

where $a_{in} = \left(\frac{\partial \Phi_i}{\partial y_n}\right) y_n = 0; R_i(y_1, y_2, ..., y_n)$ – the set of members dependent on deviations y_i in a degree above the first.

Since the deviations y_i are small enough, we neglect $R_i(y_1, y_2, ..., y_n)$ to obtain linearized equations:

$$\frac{dy_i}{dt} = a_{i1}y_1 + a_{i2}y_2 + ... + a_{in}y_n, \ i = \overline{1, n},$$

called the equations of the first approximation.

The characteristic equation of the system can be represented as follows:

$$A(p) = \begin{vmatrix} a_{11} - p & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} - p & ..... & a_{2n} \\ ...... & .... & ..... & .... \\ a_{n1} & a_{n1} & ..... & a_{nn} - p \end{vmatrix} = 0.$$

By solving the characteristic equation, we can determine its roots p_k, where $i = \overline{1, n}$, which in general have the form $p_k = \alpha_k \pm j\omega_k$, where α_k, ω_k – are real and imaginary parts respectively.

The following Lyapunov theorems are fundamentally important for the study of the stability of systems by their linearized equations.

Theorem 1. If the real parts of all roots p_k of the characteristic equation A (p) = 0 of the first approximation are negative, then the forced motion is asymptotically stable.

Theorem 2. If among the roots p_k of the characteristic equation of the first approximation A (p) = 0 there is at least one root with a positive real part, then the undisturbed motion is unstable.

It should be noted that if the heart of the roots of the characteristic equation is one or more zero koen, and the other roots have negative real parts, then the estimation of stability by the equations of the first approximation is impossible. In this case, it is necessary to consider the differential equations in their original form and use other stability criteria for nonlinear systems.

Example. Investigate the stability of the system by the first Lyapunov method, if the equations of a closed system have the following form:

$$\begin{cases} \dfrac{dy_1(t)}{dt} = y_2 - y_3; \\ \dfrac{dy_2(t)}{dt} = y_1^2 + y_2; \\ \dfrac{dy_3(t)}{dt} = y_1^2 + y_3. \end{cases}$$

The equations of the system of linear approximation around the point $y_1 = y_2 = y_3 = 0$ will look like:

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 - y_3; \\ \dfrac{dy_2}{dt} = y_2; \\ \dfrac{dy_3}{dt} = y_3. \end{cases}$$

Characteristic equation

$$A(p) = \begin{vmatrix} -p & 1 & -1 \\ 0 & 1-p & 0 \\ 0 & 0 & 1-p \end{vmatrix} = 0,$$

$$A(p) = p(1-p^2) = 0.$$

From here we get $p_1=0$, $p_2=p_3=1$.

Thus, according to the linear approximation, the undisturbed motion will be unstable, as there are positive roots.

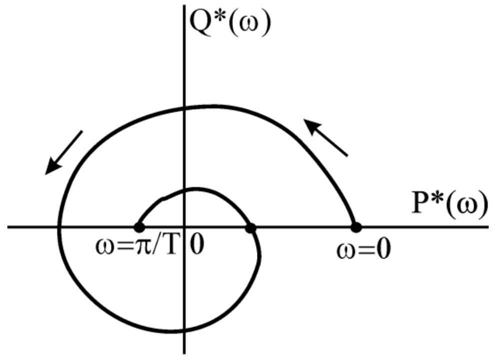

Fig. 36

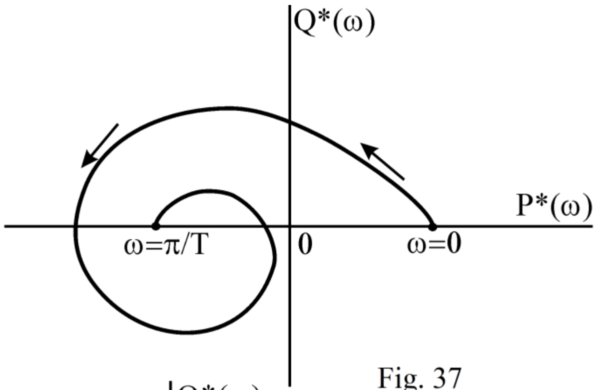

Fig. 37

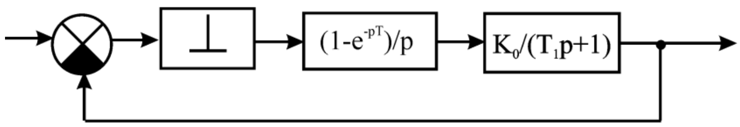

Fig. 38

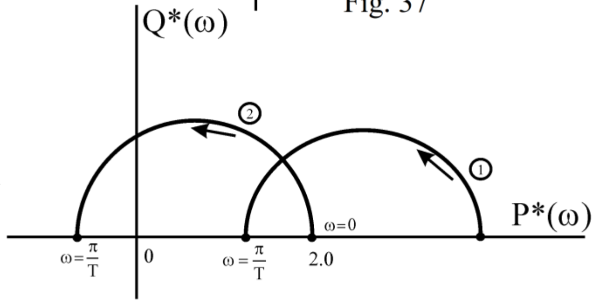

Fig. 39

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